

1. In an elastic collision of two particles with masses  $m_1$  and  $m_2$ , the initial velocities are  $\mathbf{u}_1 = \mathbf{u}$  and  $\mathbf{u}_2 = \alpha \mathbf{u}$ . If the initial kinetic energies of the two particles are equal, find the conditions on  $u_1/u_2$  and  $m_1/m_2$  such that  $m_1$  is at rest after the collision. (20 分)

2. A system consists of three particles with positions, velocities and masses as follows

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{i} + \mathbf{j}, & \mathbf{v}_1 &= 2\mathbf{i}, & m_1 &= m \\ \mathbf{r}_2 &= \mathbf{j} + \mathbf{k}, & \mathbf{v}_2 &= \mathbf{j}, & m_2 &= 2m \\ \mathbf{r}_3 &= \mathbf{k}, & \mathbf{v}_3 &= \mathbf{i} + \mathbf{j} + \mathbf{k}, & m_3 &= 3m \end{aligned}$$

(a) Find the position and velocity of the center of mass; (5 分)

(b) Find also the linear momentum of the system; (5 分)

(c) Find the angular momentum about the origin. (5 分)

3. (a) Find the curve  $y(x)$  that passes through the endpoints  $(0,0)$  and  $(1,1)$  and

$$\text{minimizes the functional } I[y] = \int_0^1 [(dy/dx)^2 - y] dx. \quad (10 \text{ 分})$$

(b) What is the minimum value of the integral? (5 分)

(c) Evaluate  $I[y]$  for a varied function  $y(\alpha, x) = x + \alpha \sin(\pi(1-x))$  and minimize  $I[y]$  with respect to  $\alpha$  to find the minimized value of  $I[y]$ . (10 分)

4. A pendulum consists of a mass  $m$  suspended by a massless spring with unextended length  $b$  and spring constant  $k$ .

(a) Find Lagrange's equations of motion, (10 分)

(b) Determine the Hamiltonian and Hamilton's equations of motion, (10 分)

5. A particle is free to slide along a smooth cycloidal trough whose surface is given by the parametric equations

$$x = \frac{a}{4}(2\theta + \sin 2\theta), \quad y = \frac{a}{4}(1 - \cos 2\theta)$$

where  $0 \leq \theta \leq \pi$  and  $a$  is a constant.

(a) Find Lagrange's equations of motion, (10 分)

(b) Determine the Hamiltonian and Hamilton's equations of motion. (10 分)